Stochastic Parametrization and Adjoint Methods:

Exploring a Toy Problem

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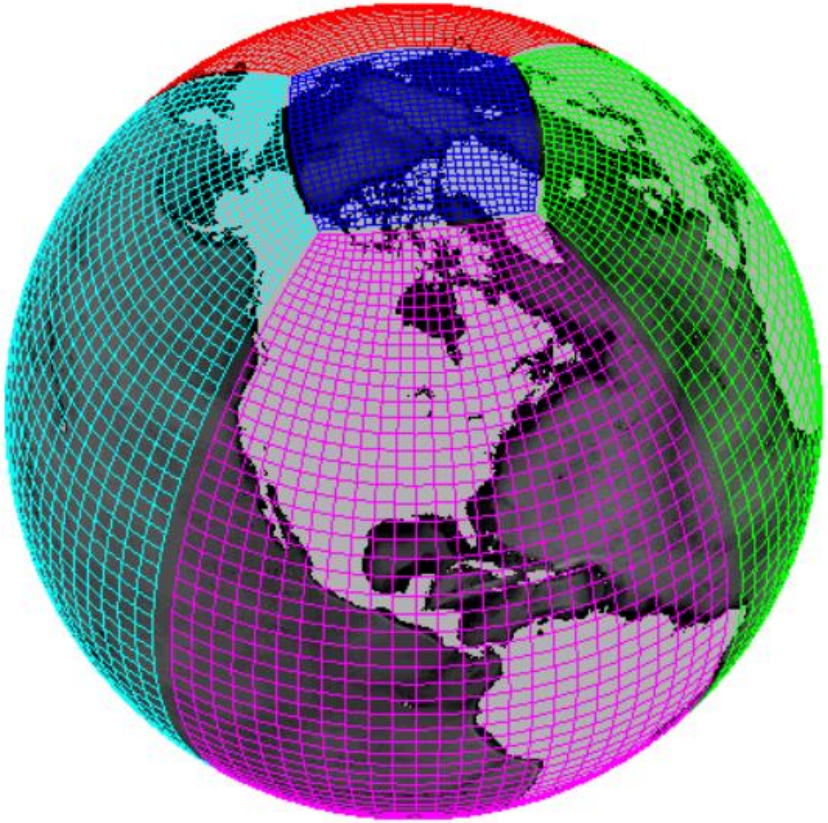
MARCH 22^{ND}, 2024

This talk

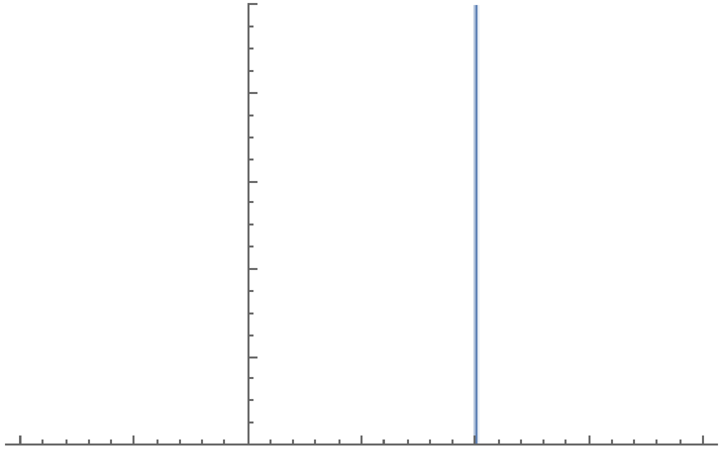
- Why stochastic parametrization?
- A toy problem: the Lorenz 96 2-scale model
- A look ahead

Subgrid-scale parametrization

- Approximate effect of the subgrid-scale on the grid-scale
- MITgcm packages:
  - gmredi: Gent-McWilliams/Redi Eddy Parameterization
  - Other mixing schemes: KPP, GGL90, KL10



Why stochastic parametrization?



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Why stochastic parametrization?

$$U_{\mathrm{det}}\left(\mathrm{params},\ \mathrm{state}\right)$$

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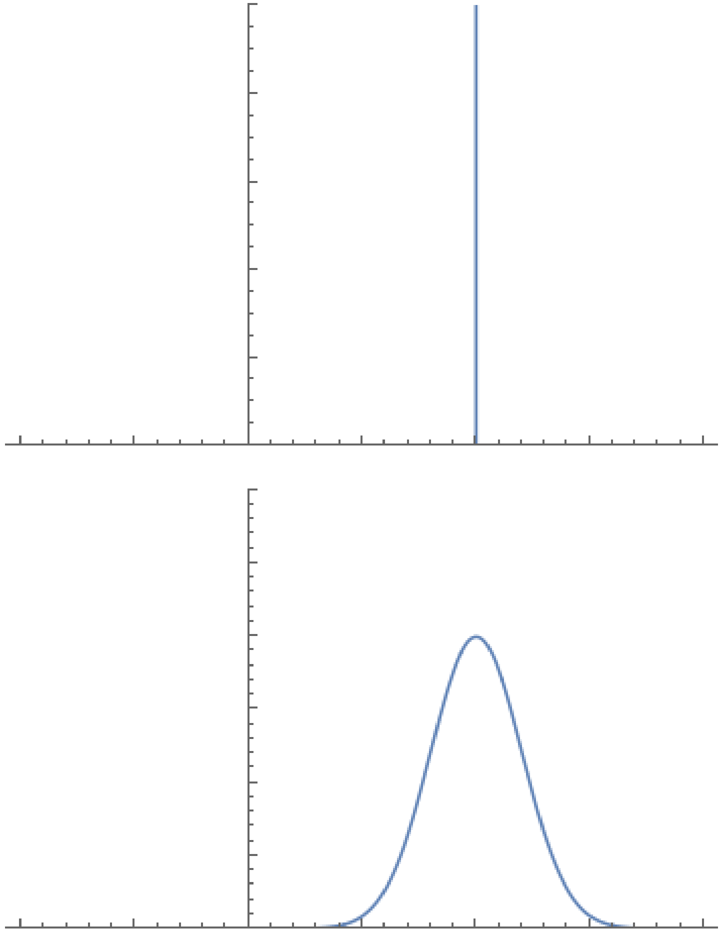

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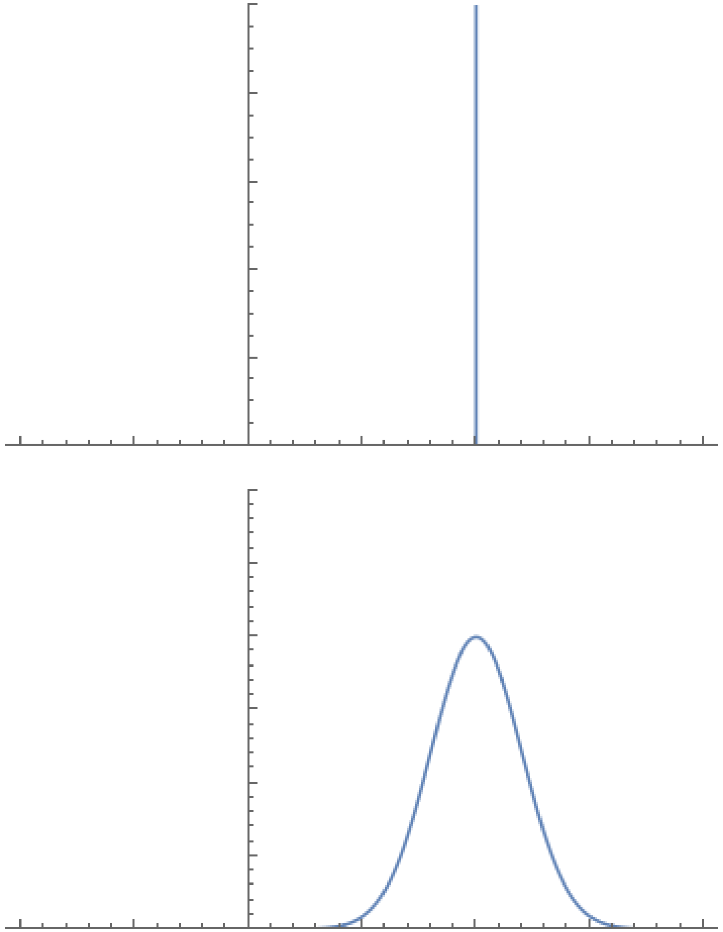

- More accurate mean model state and *variability*
  - i.e. statistical consistency of data and model
- Realistic variability ⇒ better uncertainty estimates

Lorenz 96 2-scale model

$$\frac{dX_k}{dt} = -X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F - \frac{hc}{b}\sum_{j=J(k-1)+1}^{kJ} Y_j\ ; \quad k = 1, ..., K$$

$$\frac{dY_j}{dt} = -cb\,Y_{j+1}\left(Y_{j+2} - Y_{j-1}\right) - c\,Y_j + \frac{hc}{b} X_{\mathrm{int}[(j-1)/J]+1}\ ; \quad j = 1, ..., JK$$

- *K*: number of *X* variables
- *J*: number of *Y* variables per *X* variable
- *h*: coupling (>0, sets degree of coupling)
- *F*: forcing (>0, sets degree of driving)
- *b*: spatial-scale ratio (>1 ⇒ large *X*, small *Y*)
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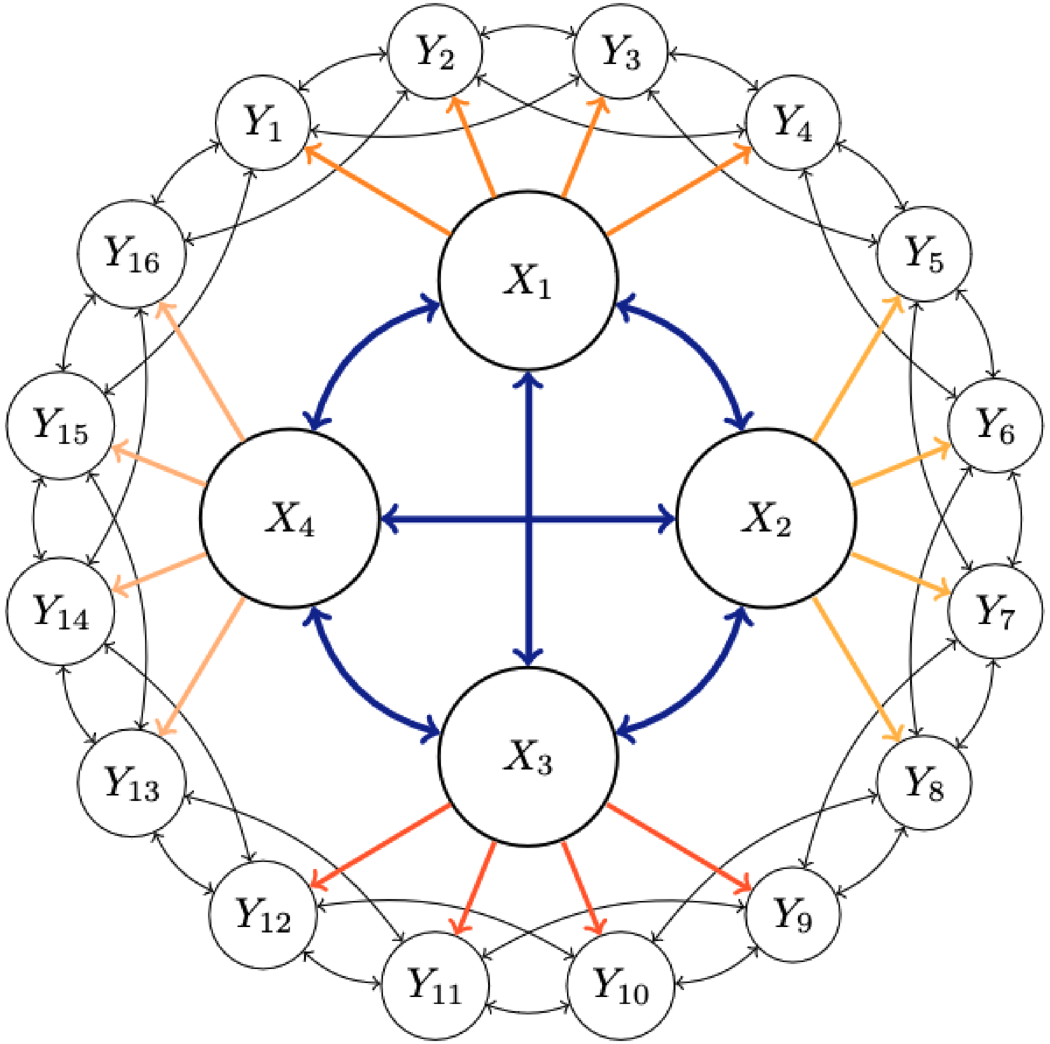
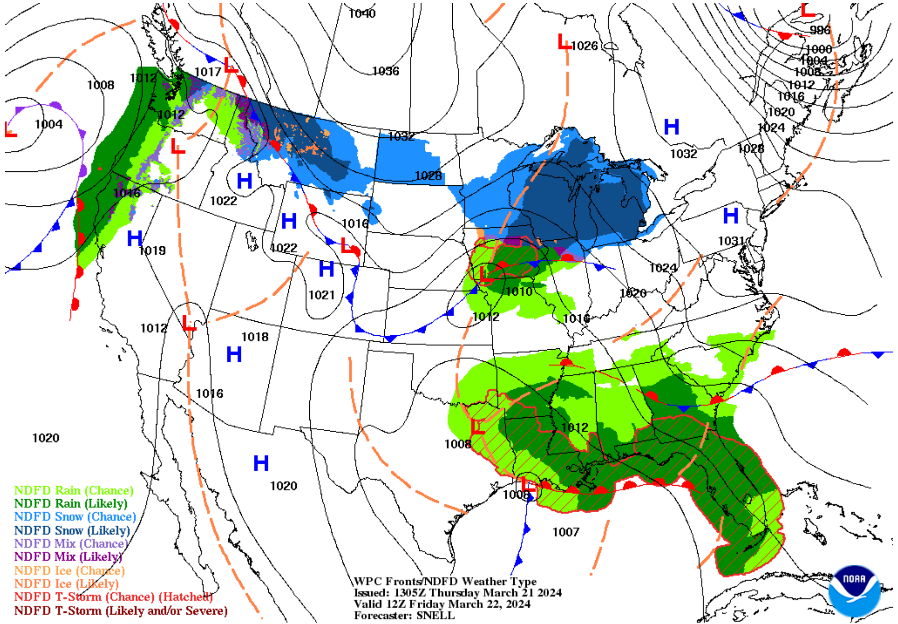
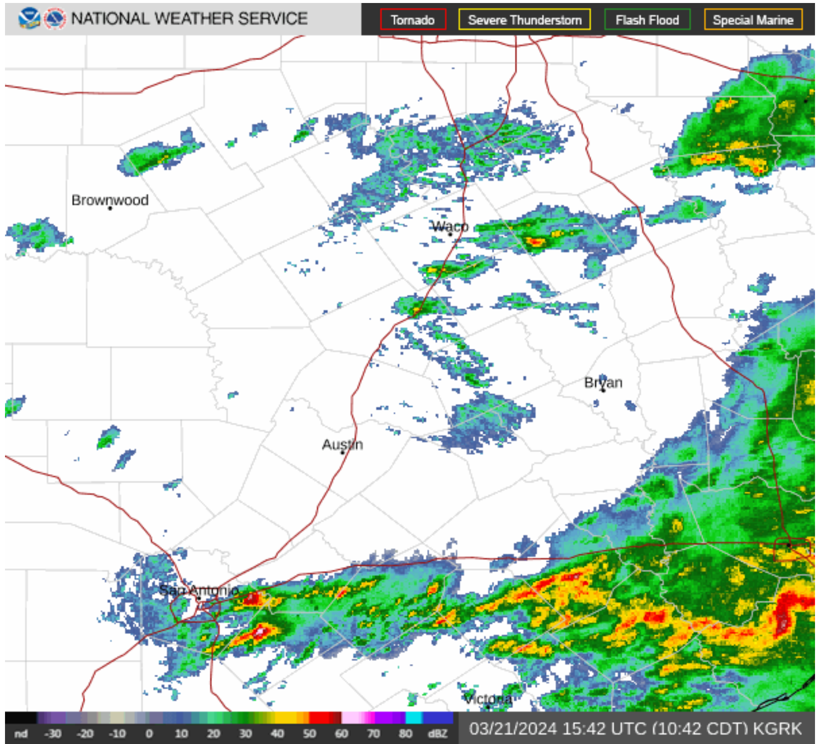


Lorenz 96 2-scale system

- Created as a toy weather model
  - *X*: large scale synoptic dynamics (e.g. low pressure system)
  - *Y*: convective events (e.g. local thunderstorms or showers)
  - Parameter choices yield chaotic *X* and *Y*





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- Could it be a proxy for ocean processes as well?

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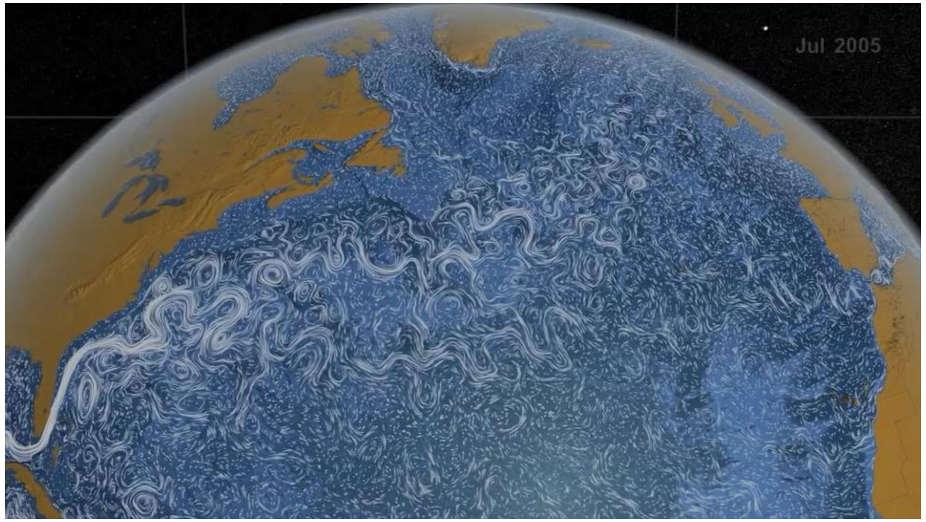

- Could it be a proxy for ocean processes as well?
  - *X*: large scale current and gyre dynamics
  - *Y*: ocean mesoscale eddies
  - Parameter choices yield **quasiperiodic** *X* and **chaotic** *Y*

- “Ocean” parameter choices:
  - $K = 4$ ⇒ 4 interacting currents
  - $J = 5$ ⇒ 5 eddy regions per current
  - $h = 2$ ⇒ Strong coupling
  - $F = 10$ ⇒ Enough to drive oscillations
  - $b = 5$ ⇒ Y amplitude 1/5 of X
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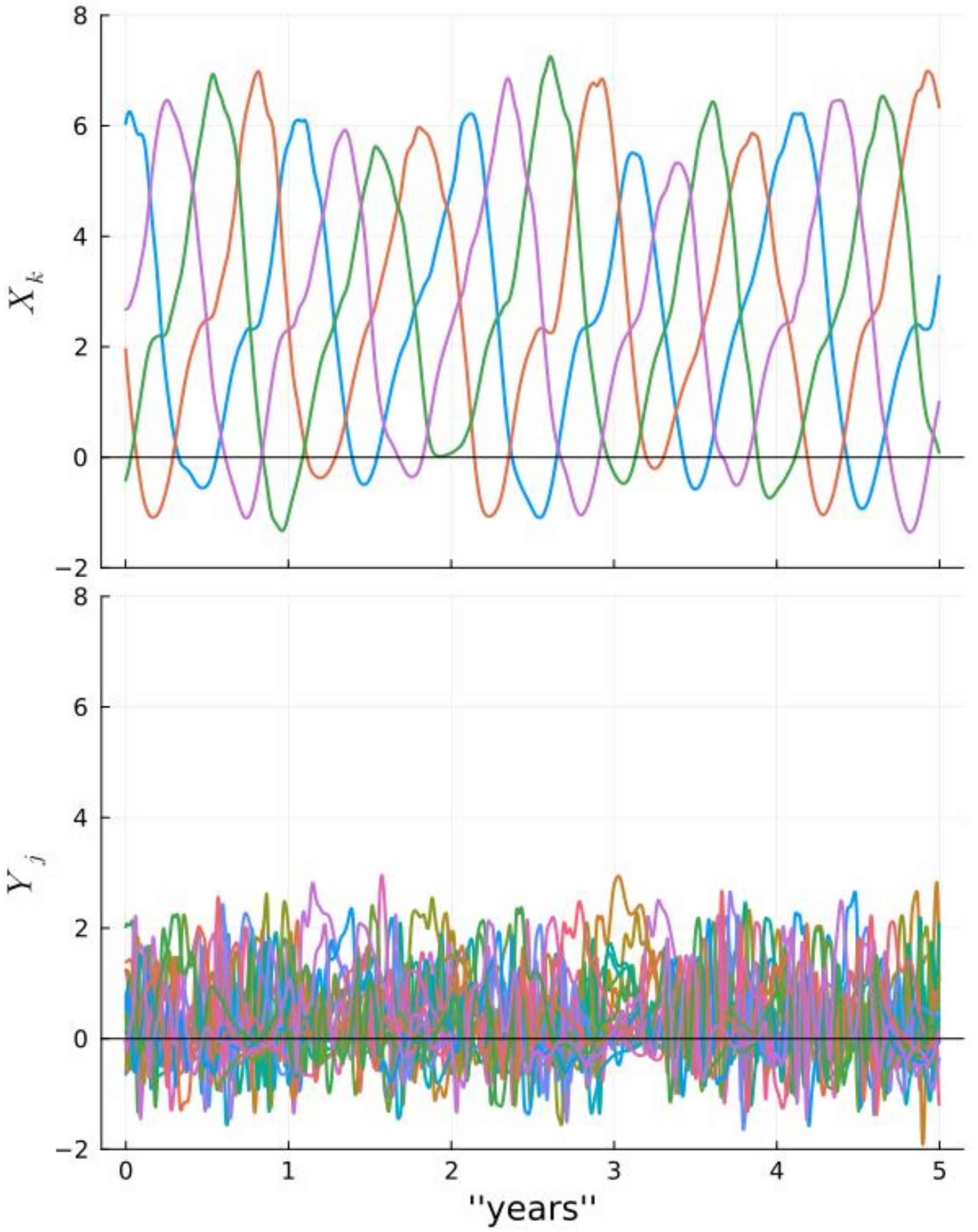
- The result?
  - Quasiperiodic oscillations in X
  - Chaos in Y



Parametrization?

- An approximate *parametrized* model of L96
  - Explicitly model only the X_k
  - Parametrize the effect of the Y_j on the X_k
  - X_k drive Y_j dynamics, which in turn affect X_k

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$$\Rightarrow \quad \frac{dX_k^*}{dt} = -X_{k-1}^*\left(X_{k-2}^* - X_{k+1}^*\right) - X_k^* + F + U_p\left(X_k^*\right) \;; \quad k = 1, ..., K$$

First guess

- Use full model “data”: least squares fit to a logistic curve

$$U_{\text{det}}\left(X^*\right) = \frac{L}{1 + e^{-k(X^* - X_0)}} + b$$

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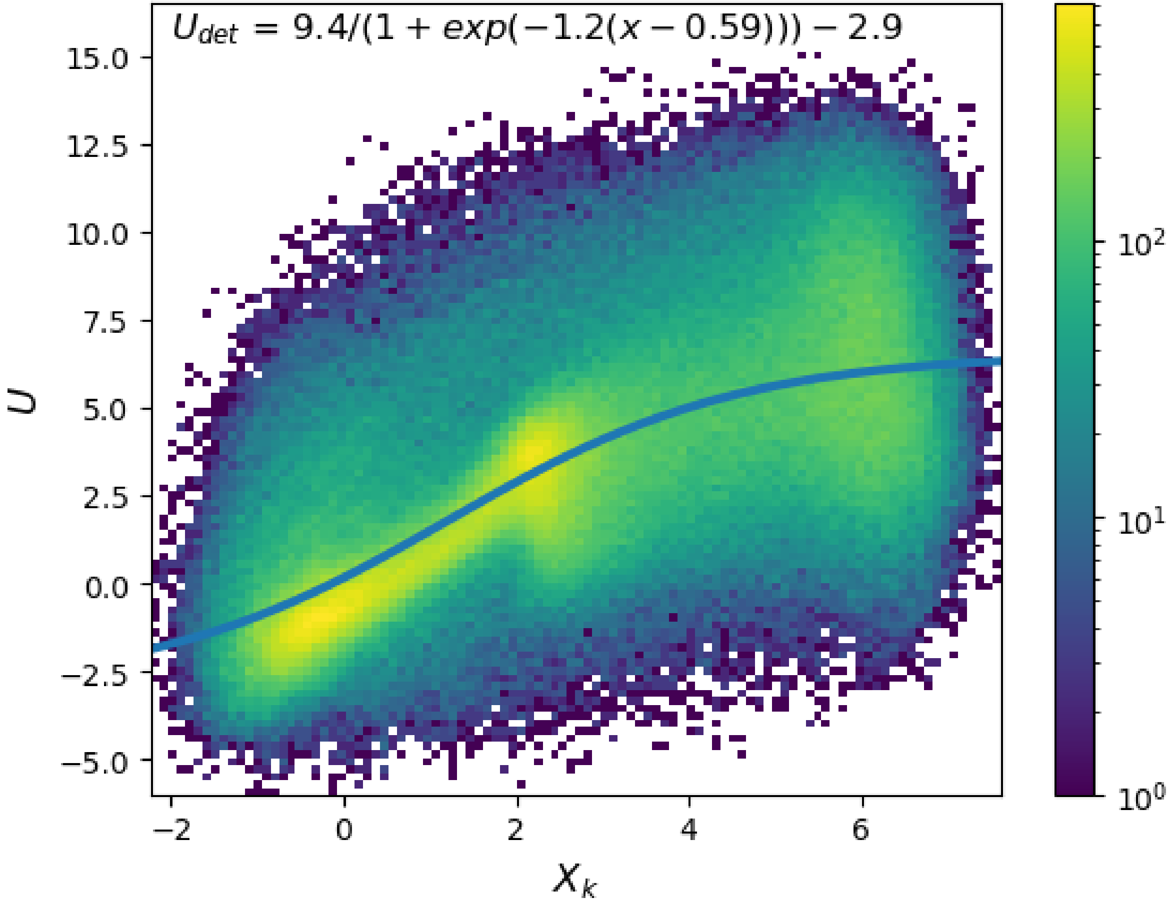
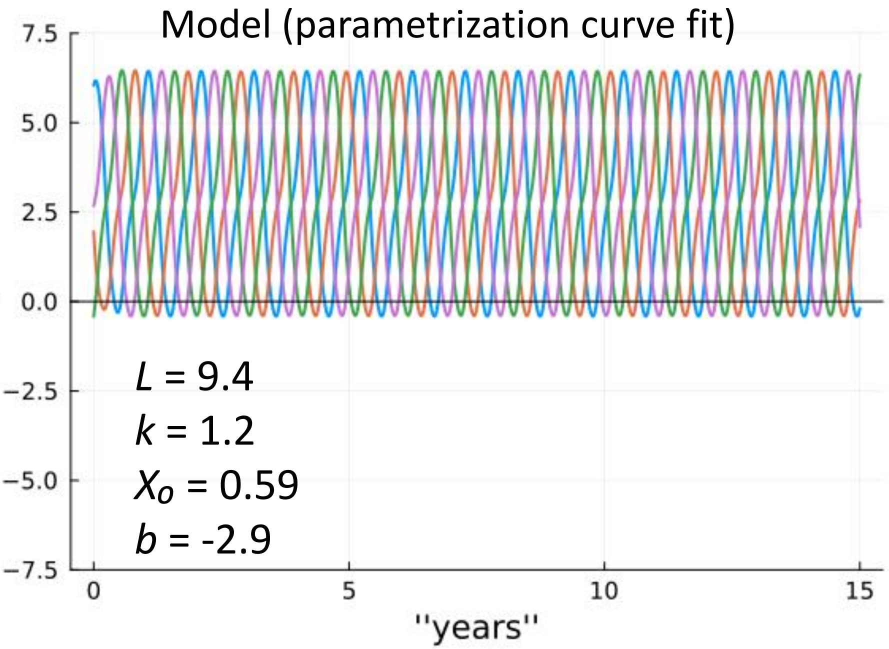


$$U_{\mathrm{det}}\left(X^*\right) = \frac{L}{1 + e^{-k(X^* - X_0)}} + b$$

- L = 9.4
- k = 1.2
- X_o = 0.59
- b = -2.9

How does it do?



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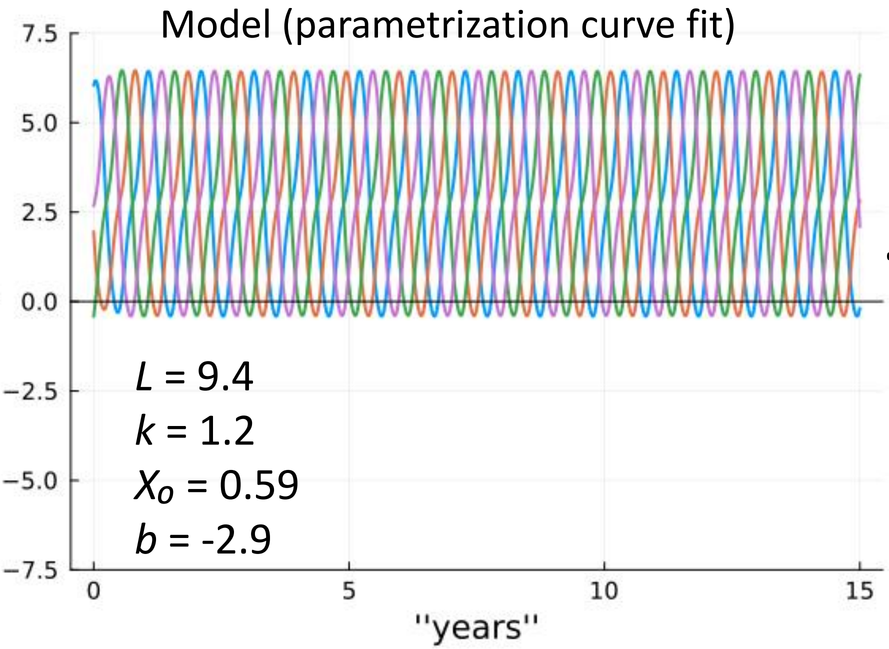

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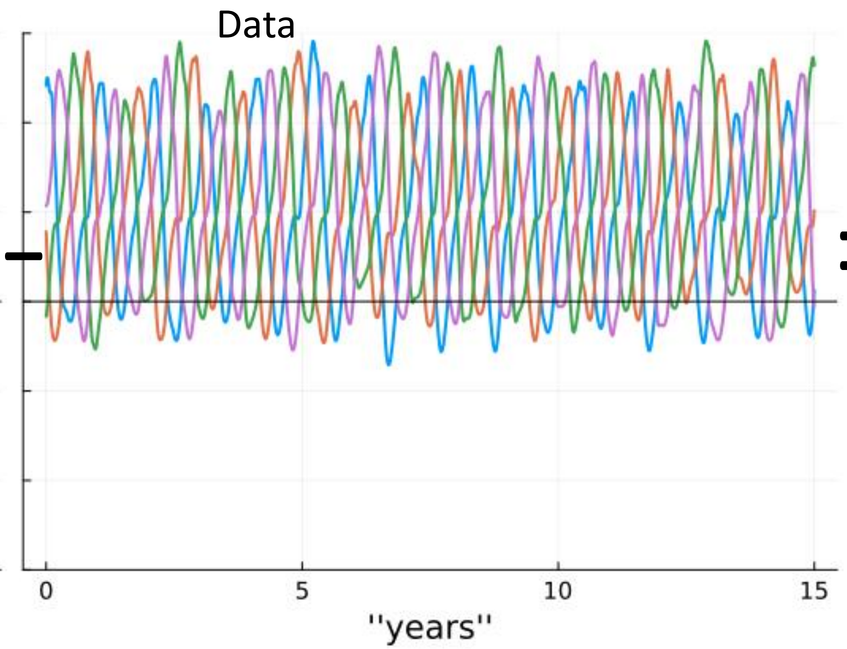

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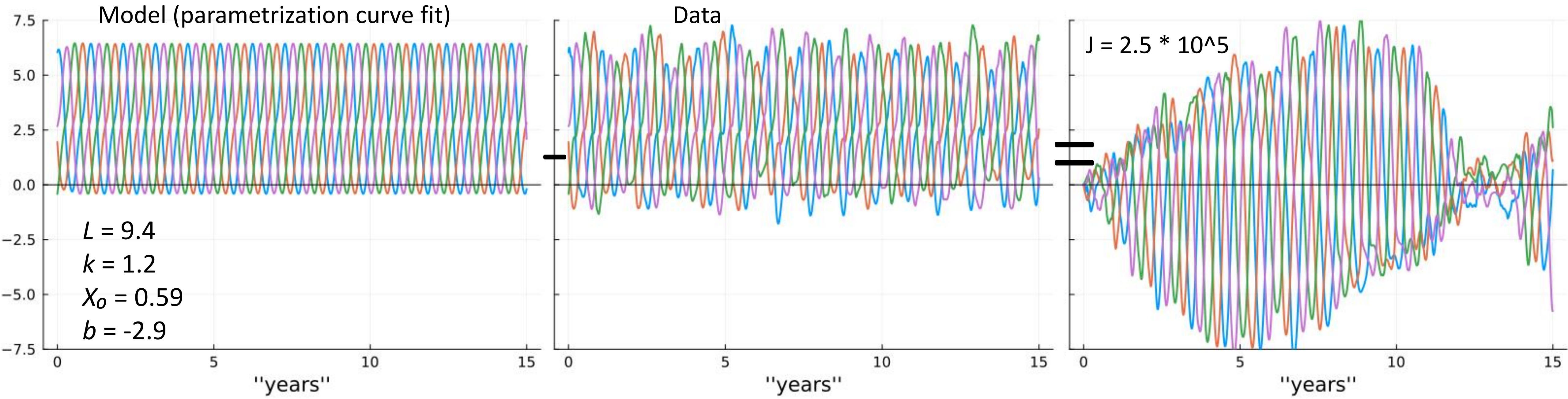

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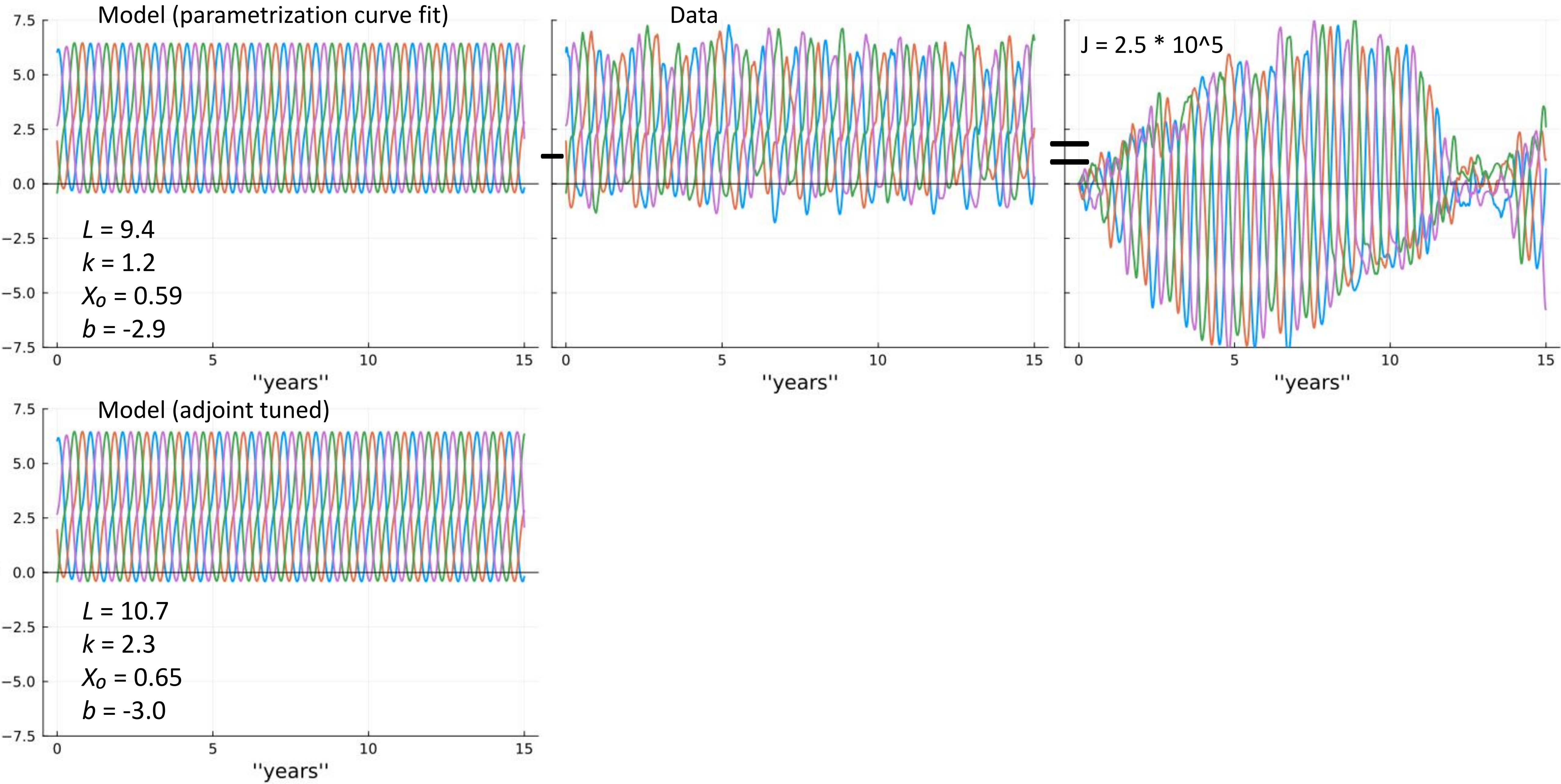

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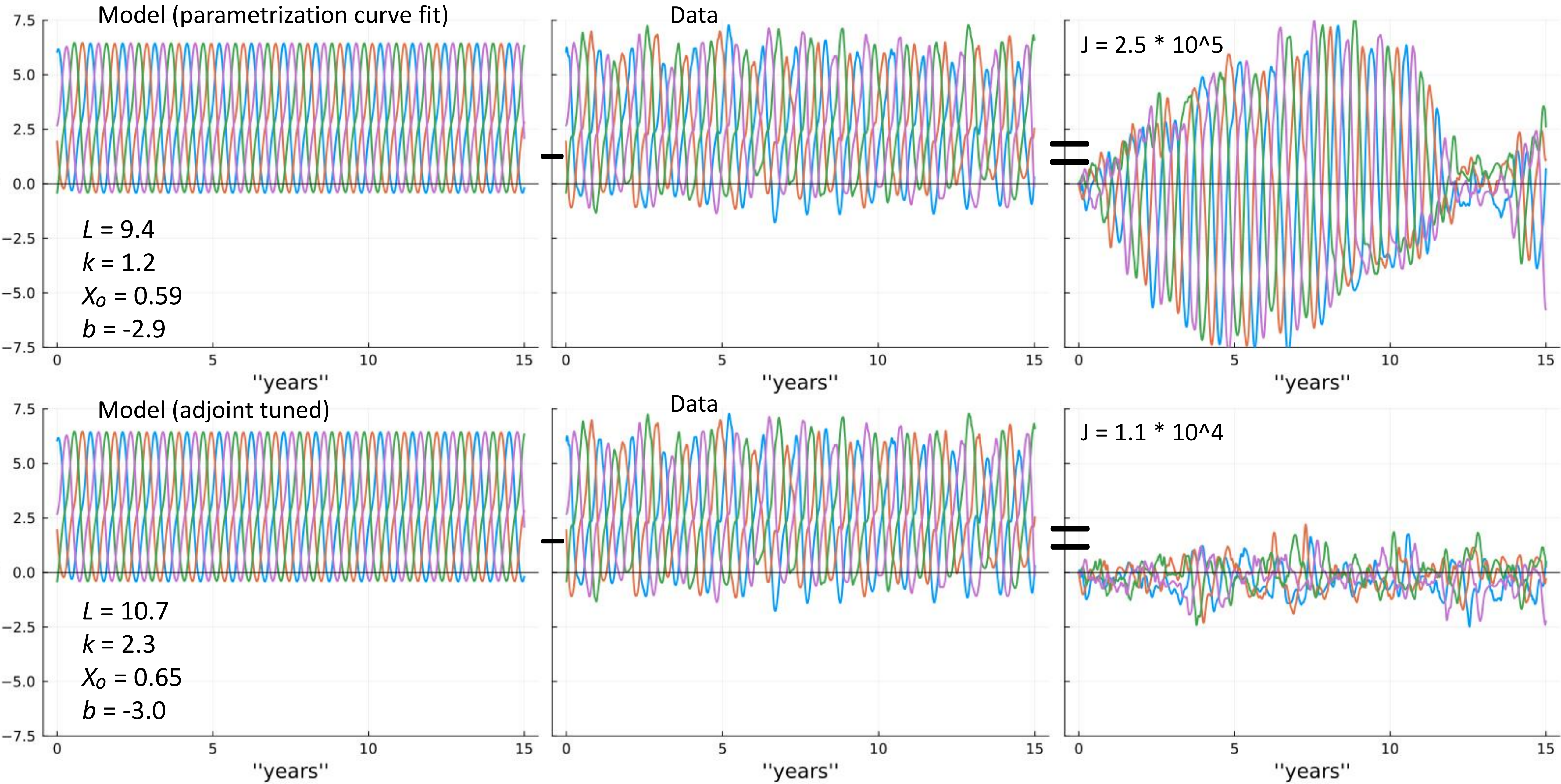

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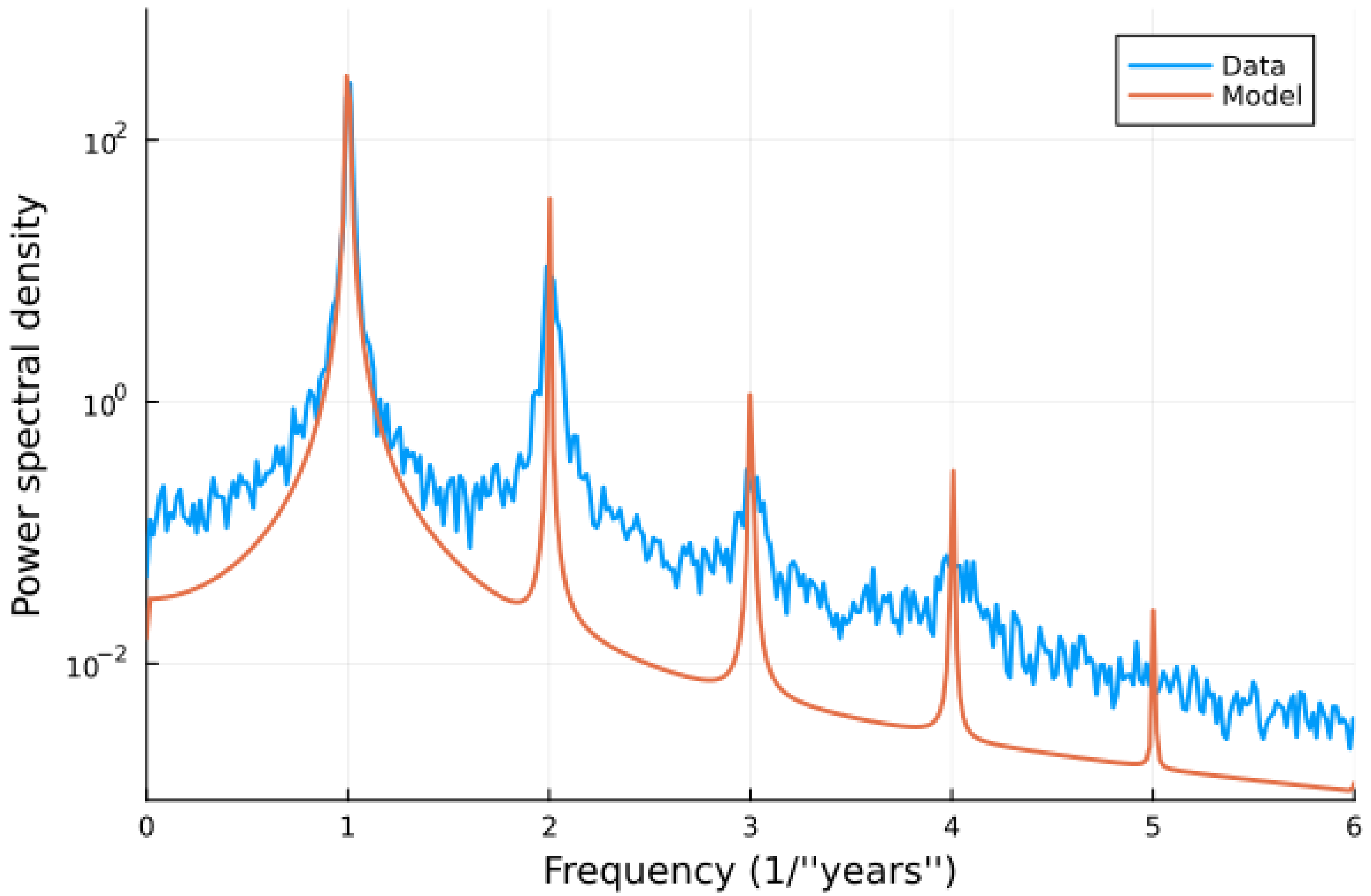

And stochastic parametrization?

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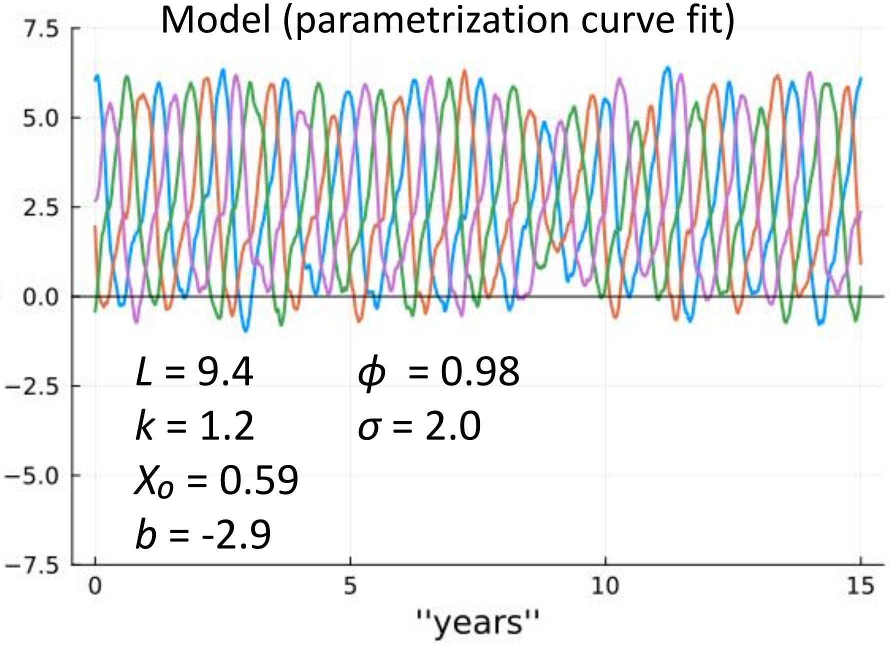
lag-1 autocorrelation

SD of *U* residual

unit variance Gaussian white noise

How about stochastic parametrization?



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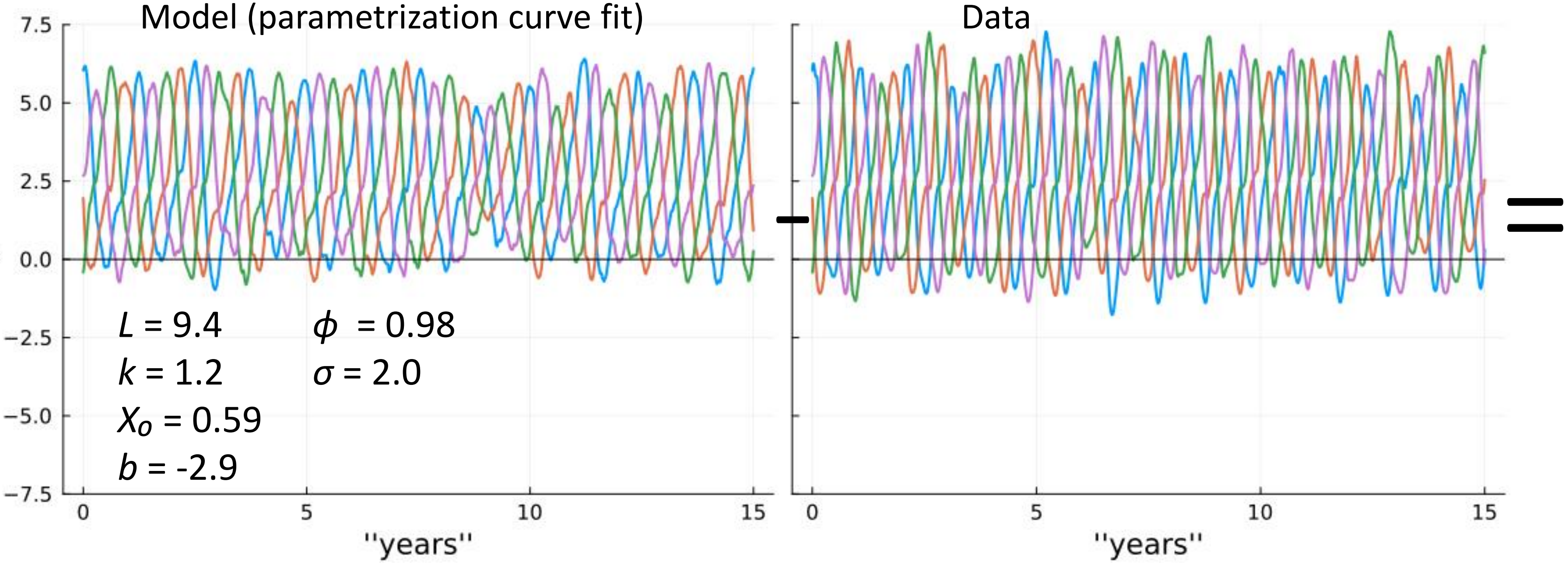

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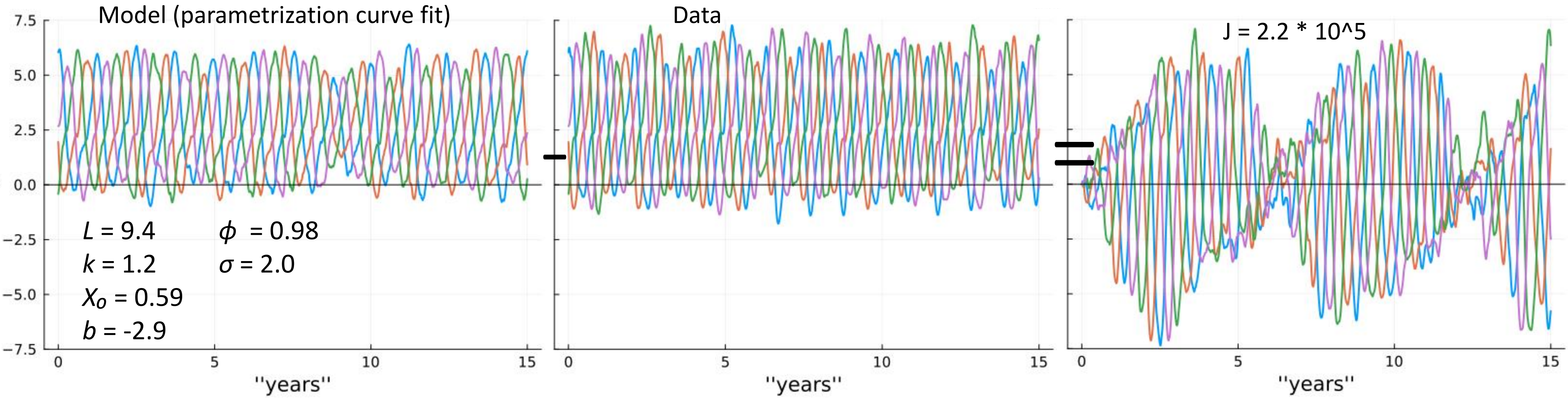

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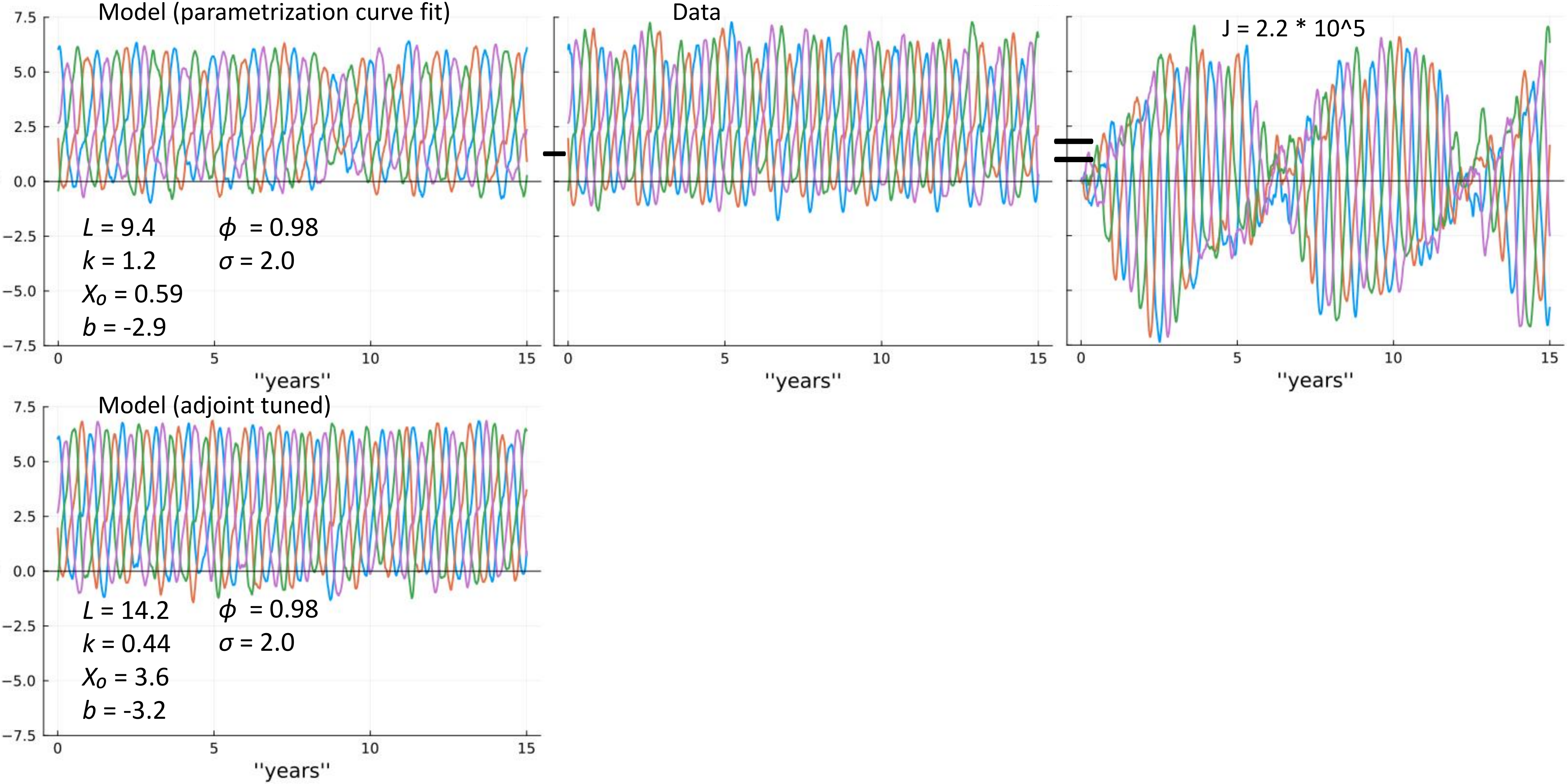

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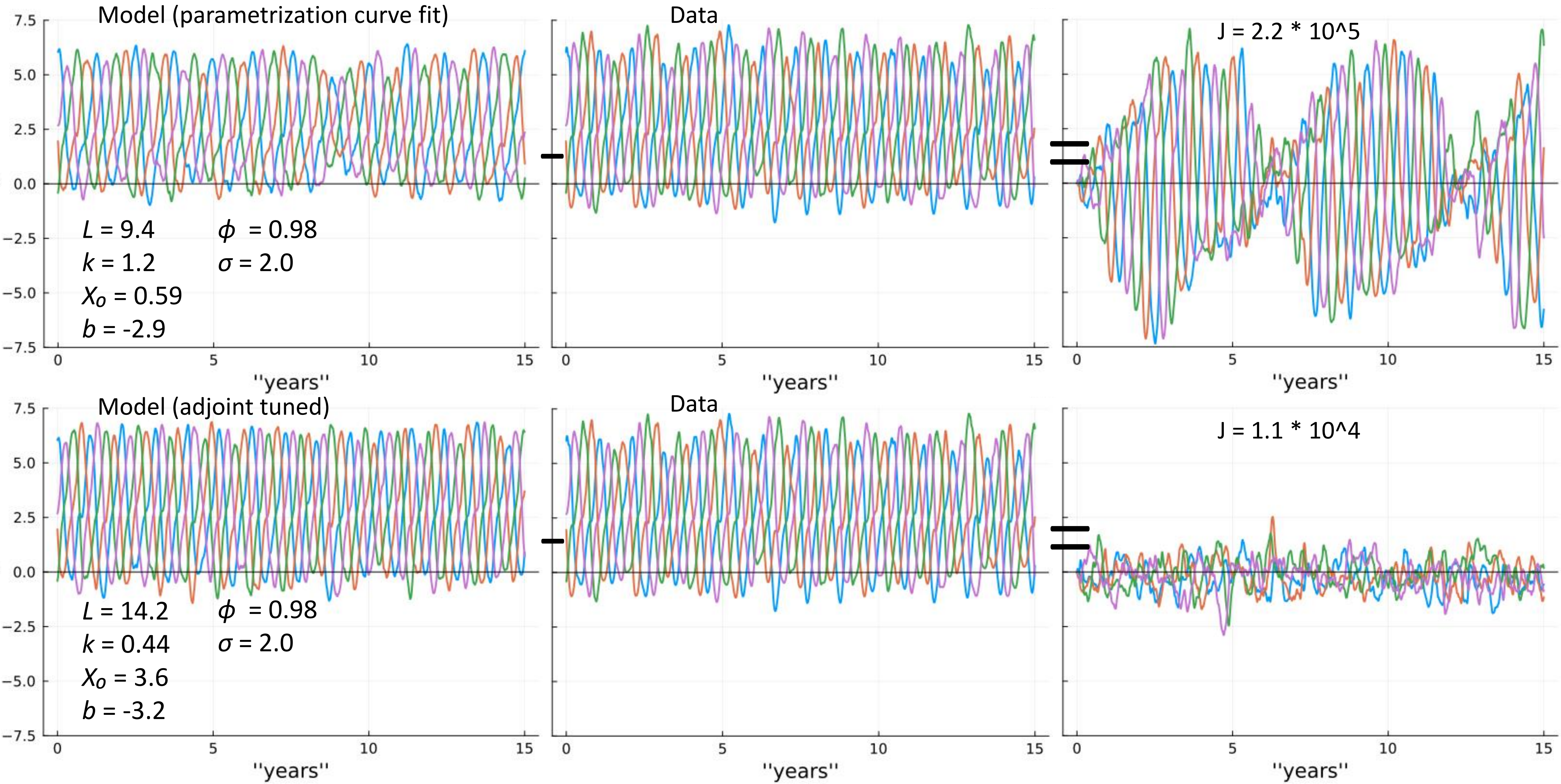

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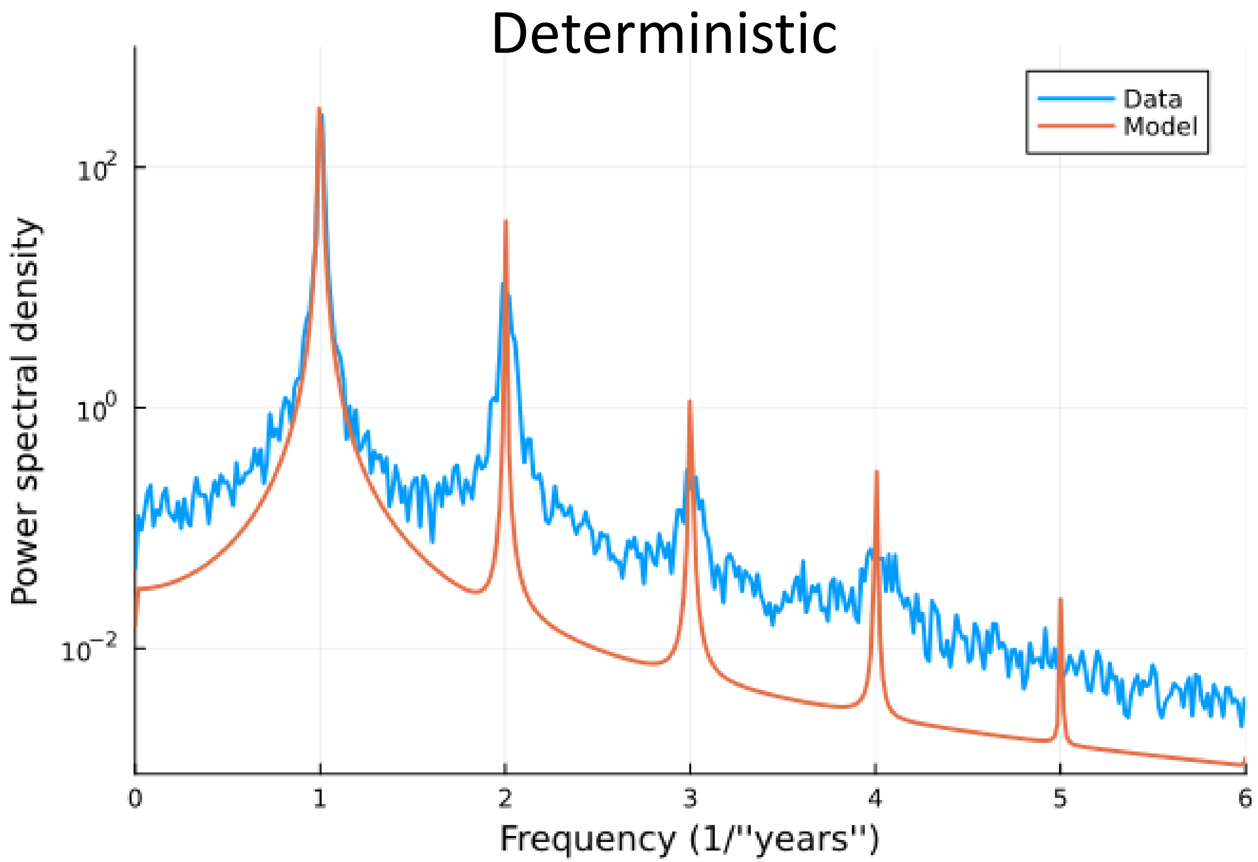

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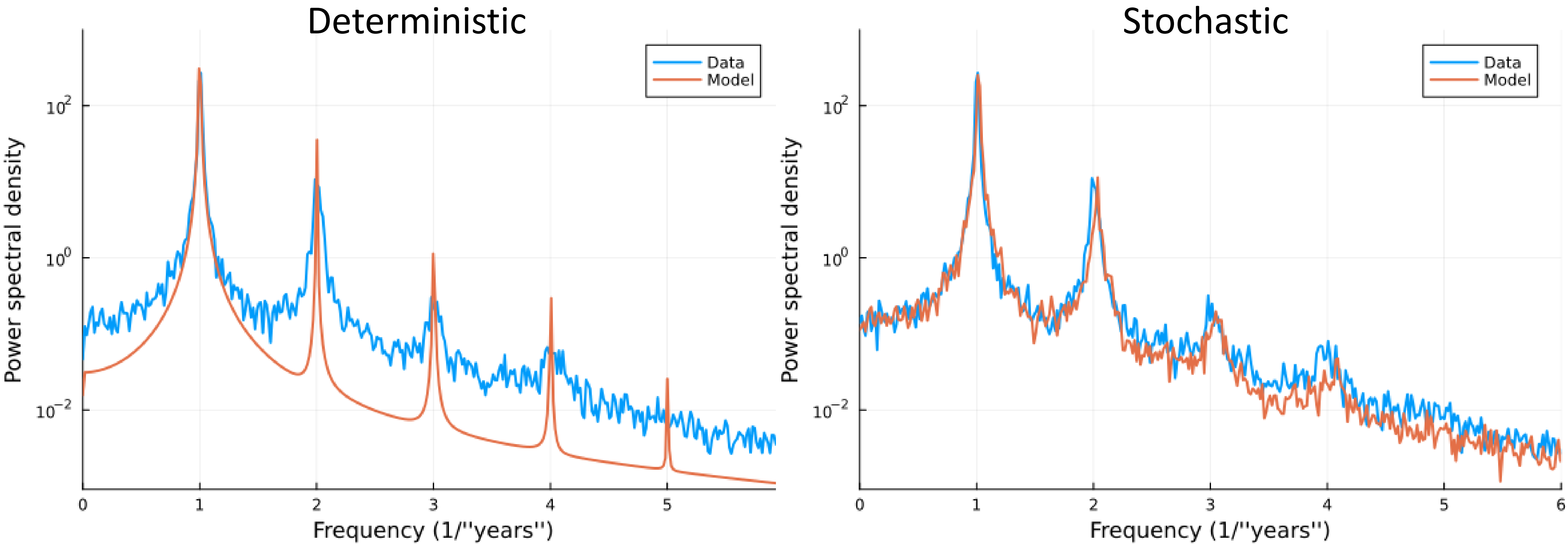

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  - Uncertainty quantification
  - Stability
  - Forecast skill
  - PSD misfit in cost function?

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  - Adjoint tuned statistics? PSD misfit in cost function?
- Idealized MITgcm setups
  - Baroclinic gyre with eddy parametrization
  - soma (MPAS-Ocean ideal model with continental shelf)